RPN81

COMPUTER APPROXIMATIONS FOR THE CONAL GREEN'S FUNCTION

by

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I. INTRODUCTION

This note provides expressions necessary for a restricted analysis of the coaxial cone diode problem. The development neglects retardation in the fields but employs correct relativistic equations for the particle motion. One is justified in doing this if one assumes the region of interest is small enough that signals propagate essentially instantaneously throughout. The reason for making this assumption is simply that the more realistic problem is too expensive of computer memory. The analysis neglects radiation as well. We can then use the two-dimensional Green's function derived previously[1]. In Section II we derive the interparticle forces for rings of charge in the conical diode. Section III is devoted to approximations for the formal expressions. The final section deals with problems of computer simulation of the time dependent flow in the diode region.

II. INTERPARTICLE FORCES IN A COAXIAL CONE DIODE

The potential at r,θ and any ϕ due to a ring of charge lying in a plane perpendicular to the axis cos θ = ±1, at r = r′ and θ = θ′ is given by[1]

$$
\begin{aligned}
V(r,\theta;r',\theta') = \frac{2\pi}{\sqrt{rr'}} \int_0^\infty \frac{\cos(\tau \log \frac{r}{r'})}{\cosh \pi\tau} \Big\{ & P_\tau(\cos\theta)\, P_\tau(-\cos\theta') \\
& - \frac{P_\tau(\cos\theta) - P_\tau(\cos\theta_1)}{P_\tau(\cos\theta_2) - P_\tau(\cos\theta_1)} P_\tau(\cos\theta_2)\Big[P_\tau(-\cos\theta') - P_\tau(-\cos\theta_1)\Big] \\
& - \frac{P_\tau(-\cos\theta') - P_\tau(-\cos\theta_2)}{P_\tau(-\cos\theta_1) - P_\tau(-\cos\theta_2)} P_\tau(-\cos\theta_1)\Big[P_\tau(-\cos\theta) - P(\cos\theta_2)\Big] \\
& - P_\tau(\cos\theta_2) P_\tau(-\cos\theta_1)\Big\} \qquad \text{for } \theta < \theta'
\end{aligned}
\tag{1}
$$

For $\theta > \theta'$, make the exchange $\theta \leftrightarrow \theta'$ in the above formula. We have written $P_\tau(\cos\theta)$ for the conal harmonic $P_{-\frac{1}{2}+i\tau}(\cos\theta)$. The conal boundaries $\theta = \theta_1$ and $\theta = \theta_2$ are defined in figure 1.

This is to be interpreted as a Coulomb Green's function; that is, its arguments are the simultaneous coordinates of the source and observer.

The charge density for a ring of charge is

$$\rho_i(r,\theta,\phi) = qQ(r,\theta,\phi)\ \delta(r-r_i)\ \delta(\theta-\theta_i)$$

where Q is chosen so that

$$q\int r^2 dr d\theta\ \sin\theta d\phi\ Q(r,\theta,\phi)\ \delta(r-r_i)\ \delta(\theta-\theta_i) = q.$$

Hence

$$\rho(r,\theta) = q\sum_{i=1}^{N} \frac{\delta(r-r_i)\ \delta(\theta-\theta_i)}{2\pi r_i^{\,2}\ \sin\theta_i}\ . \tag{2}$$

The current density is, therefore

$$\vec{j}(r,\theta) = q\sum_{i=1}^{N} \frac{v_i\delta(r-r_i)\ \delta(\theta-\theta_i)}{2\pi_i^{\,2}\ \sin\theta_i}\ . \tag{3}$$

Although we must forsake at this stage, a completely consistent relativistic treatment we will strive to retain in all field expressions all terms of order $(v/c)^2$ or less. This is accomplished by first solving for ϕ in the Coulomb gauge

$$\nabla^2\phi = -4\pi\rho$$

to get

$$\phi = \int \rho(r',\theta')V(r,\theta;r',\theta')r'^2 dr'\ \sin\theta' d\theta' d\phi'$$

$$= q\sum_i V(r,\theta;r_i,\theta_i)\ . \tag{4}$$

from which we calculate the electric field correct to zeroth order in v/c

$$\vec{E}(r,\theta) = -q\sum_i \nabla V(r,\theta;r_i\theta_i)$$

The time derivative of the electric field is, therefore,

$$\frac{\partial\vec{E}(r,\theta)}{\partial t} = -q\sum_i \nabla \frac{\partial V}{\partial t} = -q\sum_i \left(\vec{v}_i\cdot\nabla^i\right) \nabla V \left(r,\theta;r_i,\theta_i\right) \tag{5}$$

correct to first order in v/c if we neglect accelerations and hence radiation*. Now, when calculating the vector potential we include the displacement current, (5), since the convection current is itself of order v/c. Thus,

$$\vec{A} = \frac{1}{c}\int \left(\rho\vec{v} + \frac{1}{4\pi}\ \frac{\partial\vec{E}}{\partial t}\right) V d\tau \tag{6}$$

or

$$\vec{A}(r,\theta) = \frac{q}{c}\sum_i \vec{v}_i\ V(r,\theta;r_i,\theta_i)$$

$$- \frac{q}{4\pi c}\sum_i \left(\vec{v}_i\cdot\nabla^i\right) \iint \left\{ \left[\nabla' V(r',\theta';r_i,\theta_i)\right] V(r,\theta;r',\theta')\right\} 2\pi(r')^2 dr'\ \sin\theta' d\theta'$$

or, integrating the last term by parts, we have

$$\vec{A}(r,\theta) = \frac{q}{c}\sum_i \left(\vec{v}_i\ V(r,\theta;r_i,\theta_i) + \frac{q}{4\pi c}\iint \left(v\cdot\nabla^i\right) V(r',\theta';r_i\theta_i)\right.$$

$$\left.\cdot\ \nabla' V(r,\theta;r',\theta') 2\pi(r')^2 dr'\ \sin\theta' d\theta'\ \right) \tag{7}$$

since $V \equiv 0$ on the boundaries. In Appendix A we check this expression for the free space Green's function. The electric field, correct to v^2/c^2 is therefore

$$\vec{E}(r,\theta) = -q\sum_i \nabla V(r,\theta;r_i,\theta_i) - \frac{q}{c^2}\sum_i \left[\vec{v}_i(v_i\cdot\nabla^i)\ V(r,\theta;r_i\theta_i)\right.$$

$$\left. + \frac{1}{4\pi}\iint (v_i\cdot\nabla^i)^2 V(r',\theta';r_i,\theta_i)\ \nabla' V(r,\theta;r',\theta') 2\pi(r')^2 dr'\ \sin\theta' d\theta'\right]. \tag{8}$$

The only non-vanishing magnetic field component is B_ϕ:

$$B_\phi = (\nabla\times A)_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA^\theta) - \frac{\partial A^r}{\partial\theta}\right] \tag{9}$$

*by writing $\frac{\partial}{\partial t} \equiv \vec{v}_i\cdot\nabla^i$

Note that the expressions for $\vec{E}$ and B_ϕ require an additional integration to be performed if we are to consistently keep all terms to order v^2/c^2 in the force expression. This will be extremely time consuming. It also requires evaluating and storing expressions for $V(r,\theta;r',\theta')$ containing the conal functions through second degree. One may be justified in ignoring these terms initially in the interest of expediency if it can be argued that they are important only insofar as large gradients in the fields develop in the course of the problem. Eventually one will need to deal with this problem and perhaps its solution is to write V and ∇V in their power series form and integrate this expression by hand. The result could then be evaluated and stored as one has already stored V and ∇V to form the other portions of the force expressions.

The correct relativistic expressions for the particle motion can be used. This entails no extra storage since the velocities are saved for use in the field generation.

In the following section we give the approximations needed to form the fields from the conal harmonics assuming that either the integrations over the volume of the diode have been done or that they have been ignored. One needs only the expansions for harmonics of degree zero and one in this case.

III. APPROXIMATIONS FOR THE CONAL FUNCTIONS

The expression (1) requires a further integration over the degree, τ. This must be accomplished on the computer, to give a table of values for discrete choices of r and θ. It will be done more accurately if we form the expressions for the forces before the integrations on τ are performed since otherwise one would have to rely on numerical differentiation after this last integration is performed.

In general[2]

$$\frac{d}{dx} P^m_{-\frac{1}{2}+i\tau}(x) = \frac{mx}{1-x^2} P^m_{-\frac{1}{2}+i\tau}(x) - \frac{1}{\sqrt{1-x^2}} P^{m+1}_{-\frac{1}{2}+i\tau}(x) \quad .$$

Now

$$x = \pm \cos\theta, \text{ so } \frac{\partial}{\partial x} = \mp \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}$$

and since m=0, we have

$$\frac{d}{d\theta}\left[P_{-\frac{1}{2}+i\tau}(\pm\cos\theta)\right] = \pm\frac{1}{\sin^2\theta} P^1_{-\frac{1}{2}+i\tau}(\pm\cos\theta) \quad .$$

Thus we need power series approximations and asymptotic expansions for both $P^0_{-\frac{1}{2}+i\tau}(\cos\theta)$ and $P^1_{-\frac{1}{2}+i\tau}(\cos\theta)$. These are given now.[3]

For $0 \le \tau \le 2$ the following expansion is accurate to five significant figures for $0 \le \cos\theta \le 1.0$

$$P_{-\frac{1}{2}+i\tau}(\cos\theta) = \sum_{n=0}^{7} B_n \tau^{2n}$$

where

$$B_n(\theta) = \frac{2^{2n+1}}{\pi(2n)!}\int_0^{\pi/2} \frac{\left[\sin^{-1}(k\sin\xi)\right]^{2n}}{\sqrt{1-k^2\sin^2\xi}}\, d\xi$$

where $k = \sin\theta/2$.

For $P_{-\frac{1}{2}+i\tau}(-\cos\theta)$, replace θ with $\pi-\theta$. Note that

$$B_o(\theta) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\xi}{\sqrt{1-k^2\sin^2\xi}} = \frac{2}{\pi}K(\xi)$$

where K is the complete elliptic integral of the first kind

$$P^1_{-\frac{1}{2}+i\tau}(\cos\theta) = (1 + 4\tau^2)\sum_{n=0}^{7} B_n'\tau^{2n}$$

where

$$B_n'(\theta) = \frac{2^{2n+1}k^2}{\pi(2n)!\,\sin\theta}\int_0^{\pi/2} \frac{\left[\sin^{-1}(k\sin\xi)\right]^{2n}\cos^2\xi}{\sqrt{1-k^2\sin^2\xi}}\, d\xi$$

where, again, $k = \sin\frac{\theta}{2}$.

Note: $$B_o'(\theta) = \frac{2k^2}{\pi\sin\theta}\int_0^{\pi/2}\frac{\cos^2\xi\,d\xi}{\sqrt{1-k^2\sin^2\xi}} = \frac{2}{\pi\sin\theta}\left\{E(k) - (1-k^2)K(k)\right\}$$

where $E(k)$ is the complete elliptic integral of the second kind. Again for $P^1_{-\frac{1}{2}+i\tau}(-\cos\theta)$ replace θ with $\pi-\theta$.

For $\tau < 2 < 15$ ($=\infty$)

The asymptotic representations are

$$P^m_{-\frac{1}{2}+i\tau}(\cos\theta) = \frac{\tau^{m-\frac{1}{2}}}{(2\pi\sin\theta)^{\frac{1}{2}}}\,e^{\tau\theta}\left[1 - \frac{m^2-\frac{1}{4}}{2\tau}\cot\theta + O\left(\frac{1}{\tau^2}\right)\right]$$

and to get $P^m_{-\frac{1}{2}+i\tau}(-\cos\theta)$ replace θ with $\pi-\theta$. In particular,

$$P_{-\frac{1}{2}+i\tau}(\cos\theta) = \frac{e^{\tau\theta}}{(2\pi\tau\sin\theta)^{\frac{1}{2}}}\left[1 + \frac{\cot\theta}{8\tau} + O\left(\frac{1}{\tau^2}\right)\right] .$$

The asymptotic expansions are

$$P_{-\frac{1}{2}+i\tau}(\cos\theta) = \sum_{k=0}^{p-1} A_{-k}(\theta)\; I_k(\tau\theta)\,\tau^{-k} + O(\tau^{-p-\frac{1}{2}})$$

where $I_k(\tau\theta)$ is the modified Bessel function. We use the recurrence relation

$$I_{k-1}(\tau\theta) - I_{k+1}(\tau\theta) = \frac{2k}{\tau\theta}\,I_k(\tau\theta)$$

to write

$$P_{-\frac{1}{2}+i\tau}(\cos\theta) = I_o(\tau\theta)\left\{A_o + \frac{A_{-2}}{\tau^2} - \frac{4A_{-3}}{\tau^4\theta} + \cdots\right\} + I_1(\tau\theta)\left\{\frac{A_{-1}}{\tau} - \frac{2A_{-2}-\theta A_{-3}}{\tau^3\theta} + \frac{8A_{-3}}{\tau^5\theta^2} + \ldots\right\}$$

where

$$A_o = \sqrt{\frac{\theta}{\sin\theta}}$$

$$A_{-1} = \frac{1}{8\theta}\sqrt{\frac{\theta}{\sin\theta}}\,(\theta\cot\,\theta-1)$$

$$A_{-2} = \frac{1}{128\theta^2}\sqrt{\frac{\theta}{\sin\theta}}\,(9\theta^2\cot^2\theta + 6\theta\cot\theta-15 + 8\theta^2)$$

$$A_{-3} = \frac{5}{1024\theta^3}\sqrt{\frac{\theta}{\sin\theta}}\,(15\theta^3\cot^3\theta + 27\theta^2\cot\theta + 16\theta^3\cot\theta + 21\theta\cot\theta + 24\theta^2 - 63)$$

which is good for five significant figures.

$$P^1_{-\frac{1}{2}+i\tau}(\cos\,\theta) = A_1'(\theta)\tau I_1(\tau\theta) + \sum_{k=0}^{p-1} A'_{-k}(\theta)I_k(\tau\theta)\tau^{-k} + O(\tau^{-p-\frac{1}{2}})$$

$$= I_1(\tau\theta)\,\left\{A_1'\tau + \frac{A'_{-1}}{\tau} + \frac{1}{\tau^3}\left(A'_{-3} - \frac{2}{\theta}A'_{-2}\right) + \frac{8}{\tau^5\theta^2}A'_{-3} + \ldots\right\} +$$

$$+ I_o(\tau\theta)\,\left\{A_o' + \frac{1}{\tau^2}A'_{-2} - \frac{4}{\tau^4\theta}A'_{-3} + \ldots\right\}$$

where

$$A_1' = \sqrt{\frac{\theta}{\sin\theta}}$$

$$A_o' = \frac{3}{8\theta}\sqrt{\frac{\theta}{\sin\theta}}(1 - \theta\cot\theta)$$

$$A'_{-1} = \frac{1}{128\theta^2}\sqrt{\frac{\theta}{\sin\theta}}(-15\theta^2\cot^2\theta + 6\theta\cot\theta - 8\theta^2 + 9)$$

$$A'_{-2} = \frac{1}{1024\theta^3}\sqrt{\frac{\theta}{\sin\theta}}(-105\theta^3\cot^2\theta - 96\theta^3\cot\theta - 45\theta^2\cot^2\theta + 45\theta\cot\theta - 24\theta^2 + 105)$$

$$A'_{-3} = \frac{5}{2048\theta^4}\sqrt{\frac{\theta}{\sin\theta}}(-105\theta^4\cot^4\theta - 138\theta^4\cos^2\theta - 32\theta^4 - 210\theta^3\cot^3\theta - 252\theta^2\cot^2\theta$$
$$- 212\theta^3\cot\theta - 126\theta\cot\theta - 210\theta^2 + 693)$$

which is good for four significant figures.

The polynomial approximations for the modified Bessel functions are as follows[5].

Let $\tau\theta = \chi/3.75$.

$$-3.75 \le \tau\theta \le 3.75$$

Then

$$I_o(\tau\theta) = 1 + 3.5156229X^2 + 3.0899424X^4 + 1.2067492X^6 + 0.2659732X^8 + 0.0360768X^{10} + 0.0045813X^{12} + \epsilon$$

where $|\epsilon| < 1.6 \times 10^{-7}$

$$-3.75 \le \tau\theta < \infty$$

$$I_o(\tau\theta) = \frac{e^{\tau\theta}}{\sqrt{\tau\theta}} \{0.39894228 + 0.01328592X^{-1} + 0.00225319X^{-2} - 0.00157565X^{-3} + 0.00916281X^{-4} - 0.02057706X^{-5} + 0.02635537X^{-6} - 0.01647633X^{-7} + 0.00392377X^{-8} + \epsilon\}$$

where $|\epsilon| < 1.9 \times 10^{-7}$.

$$-3.75 \le \tau\theta \le 3.75$$

$$I_1(\tau\theta) = \frac{1}{\tau\theta} \{\tfrac{1}{2} + 0.87890594X^2 + 0.51498869X^4 + 0.15084934X^6 + 0.02658733X^8 + 0.00301532X^{10} + 0.00032411X^{12} + \epsilon\}$$

where $|\epsilon| < 8 \times 10^{-9}$

$$3.75 \le X < \infty$$

$$I_1(\tau\theta) = \frac{e^{\tau\theta}}{\sqrt{\tau\theta}} \{0.39894228 - 0.03988024X^{-1} - 0.00362018X^{-2} + 0.00163801X^{-3} - 0.01031555X^{-4} + 0.02282967X^{-5} - 0.02895312X^{-6} + .01787654X^{-7} - 0.00420059X^{-8} + \epsilon$$

where $|\epsilon| < 2.2 \times 10^{-7}$

When the force terms are formed from the Green's function derivatives of the Bessel functions will be created. The following two relations will be required:(5)

$$\frac{\partial I_o(z)}{\partial z} = I_1(z)$$

and

$$\frac{\partial I_1(z)}{\partial z} = zI_o(z) - I_1(z).$$

The factors $\frac{\cos(\tau \log r/r')}{\cosh \pi\tau}$ or $\frac{\sin(\tau \log r/r')}{\cosh \pi\tau}$ are no problem since for large τ, $(\cosh \pi\tau)^{-1} \rightarrow 0$ and will help the accuracy of the asymptotic expansions.

Also $$\cosh(\pi\tau) = \sum_{n=0}^{7} \frac{(\pi\tau)^{2n}}{(2n)!}$$

is accurate to 0.1% for $0 \leq \tau \leq 2$.

IV. COMPUTER SIMULATION OF THE TIME DEPENDENT PROBLEM

Generally in problems of this type we must resort to inexact methods of dealing with the interparticle interactions simply because to take account explicitly of all the forces requires inordinate computation times. Furthermore, in this case we must necessarily take this course because we have no analytic expression for the interparticle force after the machine performs the integration with respect to the degree, τ, of the conal functions.

A method for dealing with large numbers of macroparticles (typically, several thousand) which we have found to be a useful compromise between speed and accuracy is the so-called "particle-in-cell" method (PIC)(6). Imagine that the two-dimensional region between the cones in the plane ϕ=constant is divided into cells by the intersection of circles of constant r and lines of constant θ. (See figure 2) Initially the fields due to the generator pulse are the only ones present and these are calculated for only those points representing mesh points (orthogonal cell intersections). Eventually, however, the fields due to space charge must be added to the external fields. This is accomplished in the following way. One will have already created a table of values of interparticle forces where the table entries are labeled according to the position (cell coordinates) of a particle and the position of an observer, a total of four coordinates. If N_m is the total number of mesh points this table will have $N_m(N_m-1)$ entires. N_m is typically about 100. Now as one iterates the program in time the number of particles in each cell is ascertained. The force in the $\hat{r}$ and $\hat{\theta}$ directions at any given mesh point produced by the charge in any cell is simply the product of the number of particles in the cell and the appropriate tabulated values previously computed. The total force at any mesh point is the vector sum of the contributions from all cells. The actual force used to update the position and velocity of any given particle is gotten by interpolation with respect to the particles position within the cell. Because the force is, in part, velocity dependent one must also monitor the velocities of the particles in each cell at the time the particles are being counted. The particles enter the problem by appearing at the angle $\theta_2 + \delta\theta$ where θ_2 defines the cathode and $\delta\theta \ll \Delta\theta$ where $\Delta\theta$ is the angular spread delimiting a cell. They are spread randomly in r out to some R_o which is a parameter to be varied and determines the initial emission surface. If we assume that at time t = 0+ the external field has been established then particles will be produced at the cathode surface with random small velocities and will be accelerated away. If the particles are fed in rapidly enough then they will not only be drawn away by the external field but will be slowed down or driven back by the charge previously injected, those with sufficient initial velocity to get by the "virtual cathode" plane traveling on to the anode. Some particles will return the cathode and will be assumed lost. Those that reach the anode will be thereafter ignored. One can envision thereby a situation where after a time there is no net gain or loss of particles in the system and lengthly flow studies can be made by "reusing" the particles which encounter cathode or anode after injection.

The cell area should probably be kept constant over the whole diode region. This is accomplished by making the spacing in angle, $\Delta\theta$, constant and varying r according to the following

$$\int_{r_n}^{r_{n+1}} \int_{\theta}^{\theta+\Delta\theta} r dr d\theta = A$$

where A is the cell size desired. Thus

$$r_{n+1} = \sqrt{\frac{2A}{\Delta\theta} + r_n^{\,2}}$$

where A, $\Delta\theta$ and r_o are chosen upon consideration of emission radius, tube geometry and feasible number of cells.

To same time initially integration in time should probably be done via simple Euler integration. If this proves unstable one would be forced to try other techniques. Storage limitations fast become a problem in using elaborate predictor-corrector or Runge-Kutta schemes.

That storage is a problem can be appreciated by considering the following typical case:

Number of cells	100
Number of mesh points	10,000
Number of particles	2,500.

Decimal locations required:

$V(r_i, \theta_i; r_j, \theta_j)$	10,000
$\partial V/\partial\theta_i(r_i, \theta_i; r_j, \theta_j)$	10,000
$\partial V/\partial\theta_j(r_i, \theta_i; r_j, \theta_j)$	10,000
Particle coordinates $r_i, \theta_i, V_i^{\,r}, V_i^{\,\theta}$	10,000.

Additional "saved" coordinates for integration routines:

Euler	10,000
Two-point predictor-corrector	20,000
Four-point predictor-corrector	40,000.
"Left-hand side" of equations of motion	10,000
Basic Fortran Code	5 - 10,000.

It appears that one needs at least a 128K machine (e.g. IBM 360-65 or greater) for this problem.

A matter of practical interest in the coding of this problem is the question of what to do about the cathode-anode short at the common vertex of the cones. One possible way of dealing with it is to imagine a resistor connecting the electrodes at that point and superimpose the fields generated by current flow in this junction. This bias current could be a parameter of the problem.

With regard to output one is interested in observing the flow as a whole ("integration by eye") so plots of particle positions together with electrode boundaries for a sequence of times is desirable. This leads one directly to contemplate the generation of movies of this kind of output when the codes are working.

The equipotentials at any one time for many times are also of interest as are the trajectories of the particles starting at the same time for different starting places and at the same place (approximately) for different starting times. Gross tube parameters such as impedance versus time, total current vs time and voltage across the tube vs time can be compared directly with experiment.

In the future one foresees the desirability of studying the effect of ions in the diode region. These will be simply heavier and perhaps multiply charged rings which are situated in the tube before the rise of voltage from the generator (bad vacuum effect) and/or are injected into the region as the first electron rings strike the anode surface.

Another future project might be some attempt to take account of retardation, finite propagation times and radiation although this appears far from feasible now. This full relativistic treatment would provide direct comparison with the experimental problem. One would have to generate the correct TEM modes on the generator line and match them to the diode load. The current in the line would have to be compared with the tube current including the displacement current using the expression for the tube capacity. Perhaps such sophistication may be possible for other geometries for which the Green's function is given as an analytic function instead of as a table of values.

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REFERENCES

1. P. B. Ulrich, "A Two-Dimensional Green's Function for the Coaxial Cone Diode, Radiation Project Internal Report Number 12, 12 March 1969.

2. M. I. Zhurina and L. N. Karmazina, "Tables and Formulae for the Spherical Functions $P^m_{-\frac{1}{2}+i\tau}(z)$", Pergamon Press, Oxford 1966, p. 16.

3. M. I. Zhurina and L. N. Karmazina, "Asymptotic Formulae for Functions $P_{-\frac{1}{2}+i\tau}(x)$ as $\tau \rightarrow \infty$. Vychislite Pnaya matematika 6, 3-16, 1960. (I am indebted to the NRL Translator, R. S. Relac for a translation of this paper - NRL Translation Number 1167) also M. I. Zhurina and L. N. Karmazina Loc. Cit. pp 17-23.

4. M. Abramowitz and I. A. Stegun (Ed.), "Handbook of Mathematical Functions", Dover, 1965, p. 378.

5. Ibid. p. 376.

6. Francis H. Harlow, "The Particle-In-Cell Computing Method for Fluid Dynamics", in Methods of Computational Physics 3, 1964, Academic Press, B. Alder (Ed.), p. 319.

APPENDIX A

In Section II we derived an expression for the vector potential which is correct to order v/c. We show here that this is correct for the free space Green's function for which the integration can be done explicitly.

First we derive the vector potential for two free particles in Coulomb-gauge correct to order v/c.(this gives the correct field components to order v^2/c^2).

With no assumptions at all we write

$$\vec{A}(r,t) = \frac{1}{c}\int \vec{j}\ \frac{\left(\vec{r}',\ t - \frac{\vec{R}}{c}\right)}{|R|}\ d^3r'.$$

Now, to order v/c

$$f\left(r,\ t - \frac{R}{c}\right) = f(r,t) - \frac{R}{c}\frac{\partial f}{\partial t} + \ldots$$

Hence,

$$\vec{A}(r,t) = \frac{1}{c}\int \vec{j}\ \frac{(\vec{r}',t)}{|\vec{R}|}\ d^3r' - \frac{1}{c^2}\int \frac{\partial \vec{j}}{\partial t}\ (r',t)\ d^3r' .$$

For a free particle $\vec{j}(r,t) = e\vec{v}\ \delta(\vec{r} - \xi)$ so $A(r,t) = \dfrac{e\vec{v}}{c|\vec{R}|} - \dfrac{e}{c^2}\ \dfrac{\partial \vec{v}}{\partial t}$,

where

$$\vec{R} \equiv \vec{r} - \vec{\xi}(t),\ \vec{v} \equiv \frac{\partial \vec{\xi}}{\partial t} .$$

If we retain terms to order v/c we have

$$\vec{A}(\vec{r},t) = \frac{e\vec{v}}{c|\vec{R}|} .$$

To order v^2/c^2 we have, by the same arguments

$$\phi(r,t) = \frac{e}{|\vec{R}|} + \frac{e}{2c^2}\ \frac{\partial^2\ |\vec{R}|}{\partial t^2} .$$

Now to write these in Coulomb gauge we transform

$$\phi = \phi' - \frac{1}{c} \frac{\partial f}{\partial t}$$

and

$$\vec{A} = \vec{A}' + \nabla f$$

where

$$f = -\frac{e}{2c} \frac{\partial |\vec{R}|}{\partial t} = -\frac{e}{2c} \left(- \frac{\vec{v} \cdot \vec{R}}{|\vec{R}|} \right)$$

so in Coulomb gauge

$$\phi' = \frac{e}{|\vec{R}|}$$

and

$$\vec{A}' = \vec{A} - \nabla f = \frac{e\vec{v}}{|\vec{R}| c} - \frac{e}{2c} \nabla \left(\frac{\vec{v} \cdot \vec{R}}{|\vec{R}|} \right)$$

or

$$\vec{A}' = \frac{e\vec{v}}{|\vec{R}| c} - \frac{e}{2c} \left[\frac{\vec{v}}{|\vec{R}|} - \frac{1}{|\vec{R}|^3} (\vec{v} \cdot \vec{R}) \vec{R} \right]$$

so

$$\vec{A} = \frac{e}{2c} \left[\frac{\vec{v}}{|\vec{R}|} + \frac{(\vec{v} \cdot \vec{R})\vec{R}}{|\vec{R}|^3} \right] .$$

Now let us derive $\vec{A}$ from equation (7) of Section II where we use the free-space Green's function for V:

$$V(\vec{r}', \vec{r}) = \frac{1}{|\vec{r} - \vec{r}'|} .$$

Then

$$\vec{A}(\vec{r}) = \frac{e}{c} \frac{\vec{v}_1}{|\vec{r} - \vec{r}_1|} + \frac{e}{4\pi c} (\vec{v}_1 \cdot \nabla_1) \int \frac{\vec{r}' - \vec{r}_1}{|\vec{r}' - \vec{r}_1|^3} \frac{1}{|\vec{r} - \vec{r}'|} d^3\vec{r}' .$$

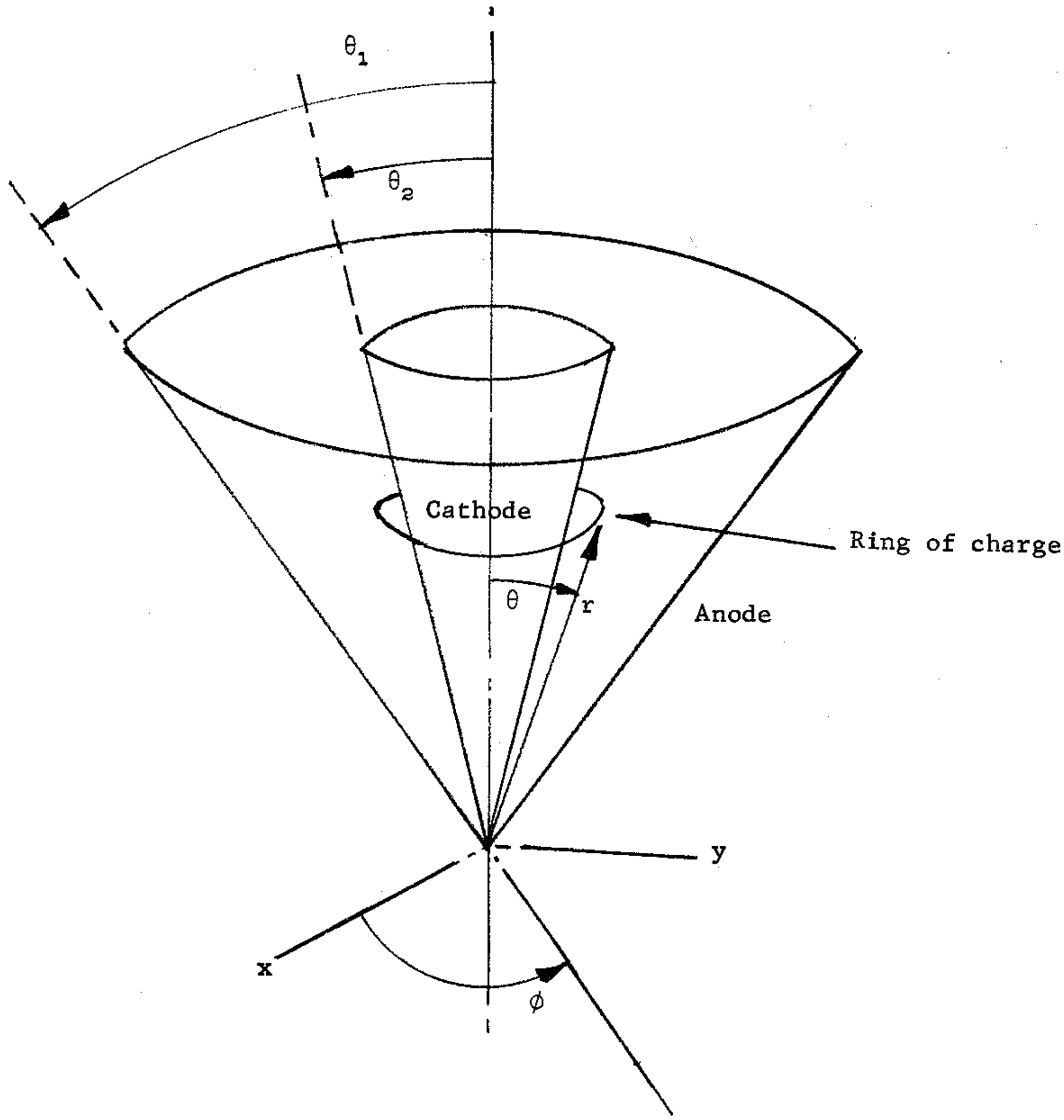
The integral is just $2\pi \dfrac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|}$

so

$$\vec{A}(\vec{r}) = \frac{e}{c} \frac{\vec{v}_1}{|\vec{r} - \vec{r}_1|} + \frac{e}{2c}\left(\vec{v}_1 \cdot \nabla_1\right) \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|} = \frac{e}{c} \frac{\vec{v}_1}{|\vec{r} - \vec{r}_1|} - \frac{e}{2c} \frac{\vec{v}_1}{|\vec{r} - \vec{r}_1|}$$

$$+ \frac{e}{2c}\left(\vec{r} - \vec{r}_1\right) \frac{\vec{v}_1 \cdot (\vec{r} - \vec{r}_1)}{|\vec{r} - \vec{r}_1|^3}$$

which agrees with the above with $\vec{r} - \vec{r}_1 \equiv \vec{R}$.



Plane section at arbitrary angle ϕ:

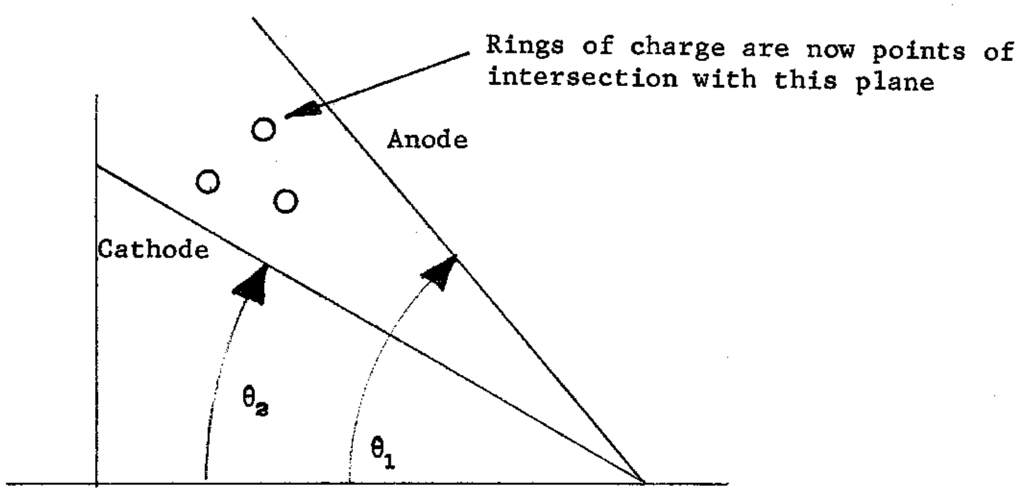


FIGURE 1

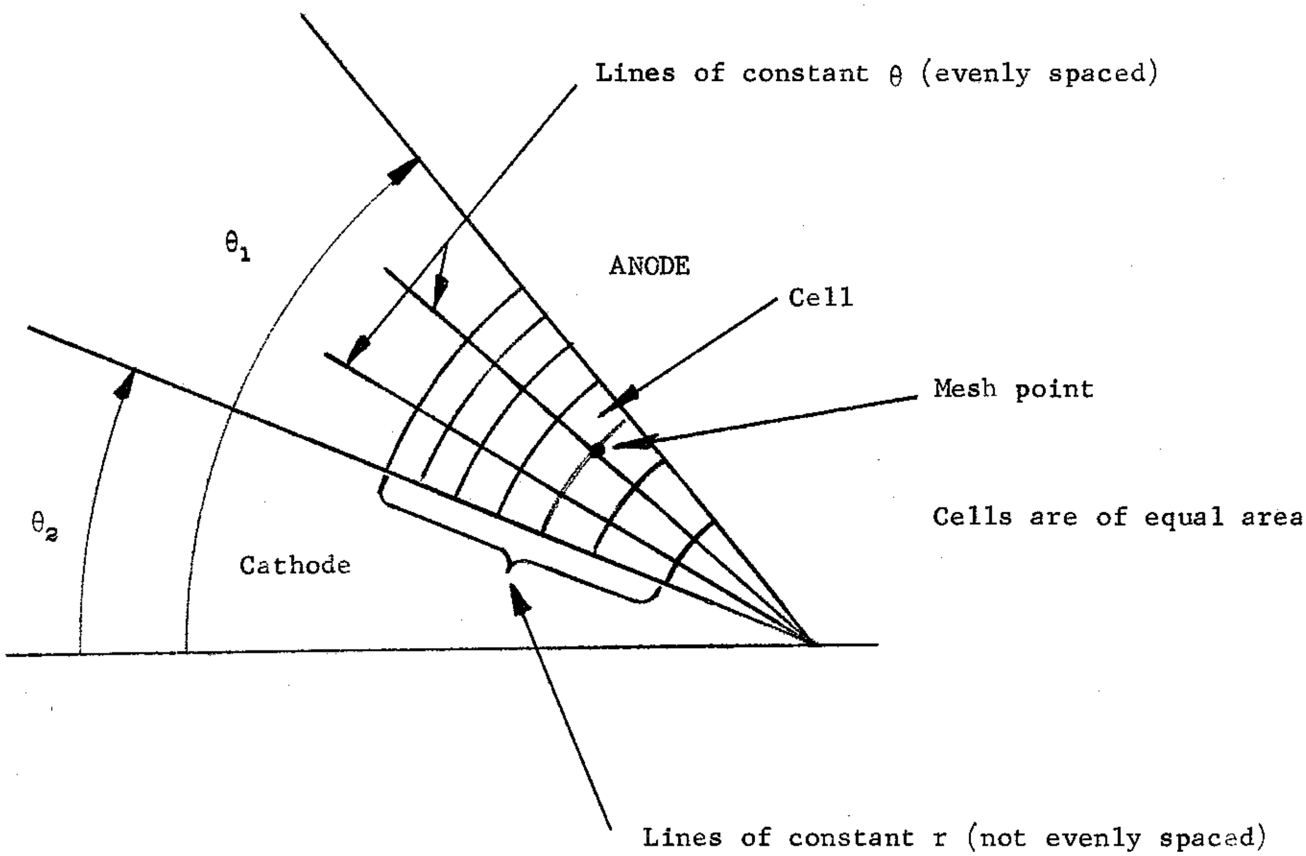

Lines of constant θ (evenly spaced)
θ1
ANODE
Cell
Mesh point
Cells are of equal area
θ2
Cathode
Lines of constant r (not evenly spaced)

FIGURE 2